Constructing N-soliton solution for the mKdV equation through constrained flows

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Abstract

Based on the factorization of soliton equations into two commuting integrable x- and t-constrained flows, we derive N-soliton solutions for mKdV equation via its x- and t-constrained flows. It shows that soliton solution for soliton equations can be constructed directly from the constrained flows.

Keywords: soliton solution, constrained flow, mKdV equation, Lax representation

1 Introduction

It is well known that there are several methods to derive the N-soliton solution of soliton equations, such as the inverse scattering method, the Hirota method, the dressing method, the Darboux transformation, etc. (see, for example, [1, 2, 3] and references therein). In present paper, we propose a method to construct N-soliton solution for mKdV equation directly through two commuting x- and t-constrained flows obtained from the factorization of mKdV equation. It was shown in [4, 5, 6, 7] that (1+1)-dimensional soliton equation can be factorized by x- and t-constrained flow which can be transformed into two commuting x- and t-finite-dimensional integrable Hamiltonian systems. The Lax representation for constrained

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flows can be deduced from the adjoint representation of the auxiliary linear problem for soliton equations [8]. By means of the Lax representation and the standard method in [9, 10, 11] we are able to introduce the separation variables for constrained flows [12]-[16] and to establish their Jacobi inversion problem [14, 15, 16]. Furthermore, the factorization of soliton equations and separability of the constrained flows allow us to find the Jacobi inversion problem for soliton equations [14, 15, 16]. By using the Jacobi inversion technique [17, 18], the N-gap solutions in term of Riemann theta functions for soliton equations can be obtained, namely, the constrained flows can be used to derive the N-gap solution. The present paper shows that the x- and t-constrained flows and their Lax representation can also be used to directly construct the N-soliton solution for soliton equations. In fact the method proposed in this paper together with that in the previous paper [19] provides a general procedure to derive N-soliton solution for soliton equations via their constrained flows.

2 The factorization of the mKdV hierarchy

We first briefly recall the constrained flows of the mKdV hierarchy and their Lax representation. The mKdV hierarchy

$$q_{t_{2n+1}} = Db_{2n+1} = D\frac{\delta H_{2n+1}}{\delta q}, \qquad n = 0, 1, \cdots, \tag{2.1}$$

with

$$H_{2n+1} = \frac{2a_{2n+2}}{2n+1},$$

is associated with the reduced AKNS spectral problem for $r = -q$ [1]

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad U = \begin{pmatrix} -\lambda & q \\ -q & \lambda \end{pmatrix}, \tag{2.2}$$

and the evolution equation of the eigenfunction

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_{t_{2n+1}} = V^{(2n+1)}(q, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.3}$$

where

$$V^{(2n+1)} = \sum_{j=0}^{2n+1} \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \lambda^{2n+1-j}, \tag{2.4}$$

with

$$a_0=-1, \quad b_0=c_0=a_1=0, \quad b_1=-c_1=q, \quad a_2=-\frac{1}{2}q^2, \quad b_2=c_2=-\frac{1}{2}q_x, \quad ...,$$

and in general

$$\begin{aligned} b_{2m+1}=-c_{2m+1}=Lb_{2m-1}, \quad L=\tfrac{1}{4}D^2+qD^{-1}qD, \quad D=\tfrac{d}{dx}, \quad DD^{-1}=D^{-1}D=1,\\ b_{2m}=c_{2m}=-\tfrac{1}{2}Db_{2m-1}, \qquad a_{2m+1}=0, \qquad a_{2m}=2D^{-1}qb_{2m}. \end{aligned} \tag{2.5}$$

For the well-known mKdV equation

$$q_t=Db_3=\frac{1}{4}(q_{xxx}+6q^2q_x), \tag{2.6}$$

the $V^{(3)}$ reads

$$V^{(3)}=\begin{pmatrix} -\lambda^3-\frac{1}{2}q^2\lambda & q\lambda^2-\frac{1}{2}q_x\lambda+\frac{1}{4}q_{xx}+\frac{1}{2}q^3 \\ -q\lambda^2-\frac{1}{2}q_x\lambda-\frac{1}{4}q_{xx}-\frac{1}{2}q^3 & \lambda^3+\frac{1}{2}q^2\lambda \end{pmatrix}. \tag{2.7}$$

We have

$$\frac{\delta\lambda}{\delta q}=\psi_1^2+\psi_2^2, \qquad L(\psi_1^2+\psi_2^2)=\lambda^2(\psi_1^2+\psi_2^2). \tag{2.8}$$

The x-constrained flows of the mKdV hierarchy consist of the equations obtained from the spectral problem (2.2) for N distinct real numbers λ_j and the restriction of the variational derivatives for the conserved quantities H_{2k_0+1} (for any fixed k_0) and $\lambda_j$defined by (see, for example, [4]-[7], [20, 21])

$$\psi_{1j,x}=-\lambda_j\psi_{1j}+q\psi_{2j}, \qquad \psi_{2j,x}=-q\psi_{1j}+\lambda_j\psi_{2j}, \qquad j=1,\cdots,N, \tag{2.9a}$$

$$\frac{\delta H_{2k_0+1}}{\delta q}-\frac{1}{2}\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta q}\equiv b_{2k_0+1}-\frac{1}{2}\sum_{j=1}^{N}(\psi_{1j}^2+\psi_{2j}^2)=0. \tag{2.9b}$$

For $k_0=0$, (2.9b) gives

$$q=\frac{1}{2}(<\Psi_1,\Psi_1>+<\Psi_2,\Psi_2>), \tag{2.10}$$

where

$$\Psi_k=(\psi_{k1},\cdots,\psi_{kN})^T, \qquad k=1,2, \qquad \Lambda=diag(\lambda_1,\cdots,\lambda_N).$$

By substituting (2.10), (2.9a) becomes a finite-dimensional integrable Hamiltonian system (FDIHS)

$$\Psi_{1x} = -\Lambda\Psi_1 + \frac{1}{2}(<\Psi_1,\Psi_1> + <\Psi_2,\Psi_2>)\Psi_2 = \frac{\partial \overline{H}_0}{\partial \Psi_2},$$

$$\Psi_{2x} = -\frac{1}{2}(<\Psi_1,\Psi_1> + <\Psi_2,\Psi_2>)\Psi_1 + \Lambda\Psi_2 = -\frac{\partial \overline{H}_0}{\partial \Psi_1}, \qquad (2.11)$$

with

$$\overline{H}_0 = -<\Lambda\Psi_1,\Psi_2> + \frac{1}{8}(<\Psi_1,\Psi_1> + <\Psi_2,\Psi_2>)^2.$$

Under the constraint (2.10), the t-constrained flow obtained from (2.3) with $V^{(3)}$ given by (2.7) for N distinct λ_j can also be written as a FDIHS

$$\Psi_{1t} = \frac{\partial \overline{H}_1}{\partial \Psi_2}, \qquad \Psi_{2t} = -\frac{\partial \overline{H}_1}{\partial \Psi_1}, \qquad (2.12)$$

with

$$\overline{H}_1 = -<\Lambda^3\Psi_1,\Psi_2> - \frac{1}{8}(<\Psi_1,\Psi_1> + <\Psi_2,\Psi_2>)^2 <\Lambda\Psi_1,\Psi_2>$$
$$+\frac{1}{4}(<\Psi_1,\Psi_1> + <\Psi_2,\Psi_2>)(<\Lambda^2\Psi_1,\Psi_1> + <\Lambda^2\Psi_2,\Psi_2>) - \frac{1}{8} <\Lambda\Psi_1,\Psi_1>^2$$
$$-\frac{1}{8} <\Lambda\Psi_2,\Psi_2>^2 + \frac{1}{4} <\Lambda\Psi_1,\Psi_1><\Lambda\Psi_2,\Psi_2> + \frac{1}{128}(<\Psi_1,\Psi_1> + <\Psi_2,\psi_2>)^4.$$

The Lax representation for the constrained flows (2.11) and (2.12), which can be obtained from the adjoint representation of the Lax representation for mKdV hierarchy [6, 8], is given by

$$M_x = [\widetilde{U}, M], \qquad M_t = [\widetilde{V}^{(3)}, M]$$

where $\widetilde{U}$ and $\widetilde{V}^{(3)}$ are obtained from U and $V^{(3)}$ by inserting (2.10) and the Lax matrix M is of the form

$$M = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}, \qquad A(\lambda) = -\lambda - \sum_{j=1}^{N} \frac{\lambda\lambda_j\psi_{1j}\psi_{2j}}{\lambda^2-\lambda_j^2},$$

$$B(\lambda) = \frac{1}{2}(<\Psi_1,\Psi_1> + <\Psi_2,\Psi_2>) + \frac{1}{2}\sum_{j=1}^{N}\frac{\lambda_j}{\lambda^2-\lambda_j^2}[(\lambda+\lambda_j)\psi_{1j}^2 - (\lambda-\lambda_j)\psi_{2j}^2],$$

$$C(\lambda) = -\frac{1}{2}(<\Psi_1,\Psi_1> + <\Psi_2,\Psi_2>) + \frac{1}{2}\sum_{j=1}^{N}\frac{\lambda_j}{\lambda^2-\lambda_j^2}[(\lambda-\lambda_j)\psi_{1j}^2 - (\lambda+\lambda_j)\psi_{2j}^2].$$

The compatibility of (2.2), (2.3) and (2.1) ensures that if Ψ_1, Ψ_2 satisfies two commuting FDIHSs (2.11) and (2.12), simultaneously, then q given by (2.10) is a solution of mKdV equation (2.6), namely, the mKdV equation (2.6) is factorized by the x-constrained flow (2.11) and t-constrained flow (2.12).

3 Constructing the N-soliton solution for the mKdV equation

Hereafter we assume that $q(x,t), \psi_{1j}, \psi_{2j}$ be real functions. For soliton solution we have $q(x,t) \to 0, \psi_{1j} \to 0, \psi_{2j} \to 0$, when $|x| \to \infty$. In order to obtain convenient formulas to construct N-soliton solution, we need to rewrite all the formulas by using the complex version instead of the vector version. We denote

$$\Phi = \Psi_1 + i\Psi_2, \qquad \phi_j = \psi_{1j} + i\psi_{2j}.$$

Then (2.11) and (2.12) become

$$\Phi_x = -\Lambda\Phi^* - \frac{i}{2}\Phi^T\Phi^*\Phi, \tag{3.1}$$

$$\Phi_t = -\Lambda^3\Phi^* - \frac{i}{2}\Phi^T\Phi^*\Lambda^2\Phi + \frac{i}{2}\Lambda\Phi^*\Phi^T\Lambda\Phi - \frac{i}{2}\Phi\Phi^T\Lambda^2\Phi^*, \tag{3.2}$$

where we have used $\overline{H}_0 = 0$.

The generating function of integrals of motion for the system (3.1) and (3.2), $\frac{1}{2}TrM^2(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda)$, gives rise to

$$A^2(\lambda) + B(\lambda)C(\lambda) = \lambda^2 - 2\overline{H}_0 + \sum_{j=1}^{N} \frac{F_j}{\lambda^2 - \lambda_j^2},$$

where $F_j, j = 1, ..., N$, are N independent integrals of motion for the systems (3.1) and (3.2)

$$F_j = 2\lambda_j^3\psi_{1j}\psi_{2j} - \frac{1}{2}\Phi^T\Phi^*\lambda_j^2(\psi_{1j}^2 + \psi_{2j}^2) + \frac{1}{4}\lambda_j^2(\psi_{1j}^2 + \psi_{2j}^2)^2 + \frac{1}{2}\sum_{k\neq j}\frac{\lambda_j^2}{\lambda_j^2 - \lambda_k^2}P_{kj},$$

$$\begin{aligned} P_{kj} = {} & \lambda_j\lambda_k(4\psi_{1j}\psi_{2j}\psi_{1k}\psi_{2k} + \psi_{1j}^2\psi_{1k}^2 + \psi_{2j}^2\psi_{2k}^2 - \psi_{1j}^2\psi_{2k}^2 - \psi_{2j}^2\psi_{1k}^2) \\ & -\lambda_k^2(\psi_{1j}^2\psi_{1k}^2 + \psi_{2j}^2\psi_{2k}^2 + \psi_{1j}^2\psi_{2k}^2 + \psi_{2j}^2\psi_{1k}^2), \quad j = 1, ..., N. \end{aligned}$$

Using (3.1), we have

$$P_{kj} = -\frac{1}{2}[\lambda_k\phi_k\phi_j^*(\lambda_k\phi_k^*\phi_j - \lambda_j\phi_k\phi_j^*) + \lambda_k\phi_j\phi_k^*(\lambda_k\phi_j^*\phi_k - \lambda_j\phi_j\phi_k^*)],$$

$$\lambda_j\phi_j\phi_j^*\partial_x^{-1}(\phi_j^2 + {\phi^*}_j^2) = -(\phi_j\phi_j^*)^2,$$

$$\lambda_k\phi_j\phi_k^* - \lambda_j\phi_k\phi_j^* = (\lambda_j^2 - \lambda_k^2)\partial_x^{-1}(\phi_j\phi_k). \tag{3.3}$$

In a similar way as we did in [19], in order to constructing N-soliton solution, we have to set $F_j = 0$. By using (3.1) and (3.3) F_j can be rewritten as

$$F_j = \frac{i}{2}\lambda_j^2\phi_j^*[-\phi_{jx} + \frac{i}{2}\sum_{k=1}^{N}\lambda_k\phi_k\partial_x^{-1}(\phi_j\phi_k)] - \frac{i}{2}\lambda_j^2\phi_j[-\phi_{jx}^* - \frac{i}{2}\sum_{k=1}^{N}\lambda_k\phi_k^*\partial_x^{-1}(\phi_j^*\phi_k^*)] = 0,$$

which leads to

$$\phi_{jx} = -\gamma_j\phi_j + \frac{i}{2}\sum_{k=1}^{N}\lambda_k\phi_k\partial_x^{-1}(\phi_j\phi_k), \qquad j = 1, ..., N,$$

or equivalently

$$\Phi_x = -\Gamma\Phi + \frac{i}{2}\partial_x^{-1}(\Phi\Phi^T)\Lambda\Phi = -\Gamma\Phi + R\Phi, \tag{3.4}$$

where $\Gamma = \mathrm{diag}(\gamma_1, ..., \gamma_N), \gamma_j$ are undetermined real numbers and

$$R = \frac{i}{2}\partial_x^{-1}(\Phi\Phi^T)\Lambda. \tag{3.5}$$

Notice that

$$\frac{i}{2}\Phi\Phi^T = R_x\Lambda^{-1}, \qquad \Lambda R = R^T\Lambda, \tag{3.6}$$

it follows from (3.4) and (3.5) that

$$R_x = \frac{i}{2}\partial_x^{-1}(\Phi_x\Phi^T + \Phi\Phi_x^T)\Lambda$$

$$= \partial_x^{-1}(-\Gamma R_x + RR_x - R_x\Gamma + R_xR) = -\Gamma R - R\Gamma + R^2. \tag{3.7}$$

We now show that $\Gamma = \Lambda$. In fact, it is found from (3.4) and (3.7) that

$$\Phi_{xx} = -\Gamma\Phi_x + R\Phi_x + R_x\Phi = -\Gamma(-\Gamma\Phi + R\Phi) + R(-\Gamma\Phi + R\Phi)$$

$$+(-\Gamma R - R\Gamma + R^2)\Phi = \Gamma^2\Phi + 2R_x\Phi = \Gamma^2\Phi + i\Phi\Phi^T\Lambda\Phi.$$

On the other hand (3.1) yields

$$\Phi_{xx} = \Lambda^2\Phi + i\Phi\Phi^T\Lambda\Phi,$$

which implies $\Gamma = \Lambda$. Therefore we have

$$\Phi_x = -\Lambda\Phi + R\Phi, \tag{3.8}$$

$$R_x = \frac{i}{2}\Phi\Phi^T\Lambda = -\Lambda R - R\Lambda + R^2. \tag{3.9}$$

To solve (3.8), we first consider the linear system

$$\Psi_x = -\Lambda\Psi.$$

It is easy to see that

$$\Psi = (\alpha_1(t)e^{-\lambda_1 x}, ..., \alpha_N(t)e^{-\lambda_N x})^T.$$

Take the solution of (3.8) to be of the form

$$\Phi = (I - M)\Psi, \tag{3.10}$$

then M has to satisfy

$$M_x = M\Lambda - \Lambda M - R + RM. \tag{3.11}$$

Comparing (3.11) with (3.9), one finds

$$M = \frac{1}{2}R\Lambda^{-1} = \frac{i}{4}\partial_x^{-1}(\Phi\Phi^T). \tag{3.12}$$

Equation (3.10) implies that

$$\Psi = \sum_{n=0}^{\infty} M^n\Phi. \tag{3.13}$$

By using (3.12) and (3.13), it is found from that

$$\frac{i}{4}\partial_x^{-1}(\Psi\Psi^T) = \frac{i}{4}\partial_x^{-1}\sum_{n=0}^{\infty}\sum_{l=0}^{n} M^l\Phi\Phi^T M^{n-l}$$

$$= \partial_x^{-1}\sum_{n=0}^{\infty}\sum_{l=0}^{n} M^l M_x M^{n-l} = \sum_{n=1}^{\infty} M^n.$$

Set

$$V = (V_{kj}) = \frac{i}{4}\partial_x^{-1}(\Psi\Psi^T), \qquad V_{kj} = -\frac{i}{4}\frac{\alpha_k(t)\alpha_j(t)}{\lambda_k + \lambda_j}e^{-(\lambda_k+\lambda_j)x},$$

one obtain

$$(I + V)\Phi = \Psi, \qquad \text{or} \qquad \Phi = (I - M)\Psi = (I + V)^{-1}\Psi. \tag{3.14}$$

Notice that (3.1) and (3.8) gives rise to

$$\Lambda\Phi^* = (\Lambda - R - \frac{i}{2}q)\Phi. \tag{3.15}$$

By inserting (3.9) and (3.15), (3.2) reduces to

$$\Phi_t = [-\Lambda^2(\Lambda - R - \frac{i}{2}q) - \frac{i}{2}q\Lambda^2 + (\Lambda - R - \frac{i}{2}q)(-\Lambda R - R\Lambda + R^2)$$

$$-(-\Lambda R - R\Lambda + R^2)(\Lambda - R - \frac{i}{2}q)]\Phi = -\Lambda^3\Phi + R\Lambda^2\Phi. \tag{3.16}$$

Let Ψ satisfy the linear system

$$\Psi_t = -\Lambda^3\Psi, \tag{3.17}$$

then

$$\Psi = (\alpha_1(t)e^{-\lambda_1 x}, ..., \alpha_N(t)e^{-\lambda_N x})^T, \qquad \alpha_i(t) = \beta_j e^{-\lambda_j^3 t}, \quad j = 1, ..., N. \tag{3.18}$$

We now show that Φ determined by (3.14) and (3.18) satisfy (3.16). In fact, we have

$$\Phi_t = -(I+V)^{-1}\frac{i}{4}\partial_x^{-1}(\Psi_t\Psi^T + \Psi\Psi_t^T)(I+V)^{-1}\Psi + (I+V)^{-1}\Psi_t$$

$$= (1-M)(\Lambda^3 V + V\Lambda^3)\Phi - (1-M)\Lambda^3(1+V)\Phi = -\Lambda^3\Phi + (I-M)V\Lambda^3\Phi + M\Lambda^3\Phi$$

$$= -\Lambda^3\Phi + 2M\Lambda^3\Phi = -\Lambda^3\Phi + R\Lambda^2\Phi.$$

Therefore Φ given by (3.14) and (3.18) satisfy (3.1) and (3.2), simultaneously, and $q = \Phi^T\Phi^*$ is the solution of mKdV equation (2.6). Notice that

$$\partial_x(\Psi^T\Phi) = -\Psi^T\Lambda\Phi + \Psi^T(-\Lambda + R)\Phi$$

$$= \Psi^T(-2I + 2M)\Lambda\Phi = -2\Phi^T\Lambda\Phi,$$

$$q_x = \frac{1}{2}(\Phi_x^T\Phi^* + \Phi^T\Phi_x^*) = \frac{1}{2}[(-\Phi^{*T}\Lambda - \frac{i}{2}q\Phi^T)\Phi^* + \Phi^T(-\Lambda\Phi + \frac{i}{2}q\Phi^*)]$$

$$= -\frac{1}{2}(\Phi^{*T}\Lambda\Phi^* + \Phi^T\Lambda\Phi) = -\mathrm{Re}(\Phi^T\Lambda\Phi).$$

So we have

$$q = \frac{1}{2}\mathrm{Re}(\Psi^T\Phi) = \frac{1}{2}\mathrm{Re}\sum_{k=1}^{N}\alpha_k(t)e^{-\lambda_k x}\phi_k. \tag{3.19}$$

Finally, as pointed out in [1], formulas (3.14) and (3.19) gives rise to the well-known N-soliton solution of mKdV equation (2.6)

$$u = 2\partial_x \mathrm{Imln}(\det(I+V)).$$

4 Conclusion

We first factorize the mKdV equation into two commuting integrable x- and t-constrained flows, then use them and their Lax representation to directly derive the N-soliton solution for mKdV equation. The method proposed in present paper and previous paper [19] provides a general procedure to construct N-soliton solution for soliton equations via their x- and t-constrained flows and can be applied to other soliton eqqquations.

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References

[1] Ablowitz M and Segur H, 1981, Solitons and the inverse scattering transform (Philadelphia, PA:SIAM).

[2] Newell A C, 1985, Soliton in Mathematics and Physics (Philadelphia, PA:SIAM).

[3] Faddeev L D and Takjtajan L A, 1987, Hamiltonian methods in the theory of solitons (Berlin: Springer).

[4] Cao Cewen, 1991, Acta math. Sinica, New Series 17, 216.

[5] Zeng Yunbo, 1991, Phys. Lett. A 160, 541.

[6] Zeng Yunbo, 1994, Physica D 73, 171.

[7] Ma W X and Strampp W, 1994, Phys. Lett. A 185, 277.

[8] Zeng Yunbo and Li Yishen, 1993, J. Phys. A 26, L273.

[9] Sklyanin E K, 1989, J. Soviet. Math. 47, 2473.

[10] Kuznetsov V B, 1992, J. Math. Phys. 33, 3240.

[11] Sklyanin E K, 1995, Theor. Phys. Suppl. 118, 35.

[12] Eilbeck J C, Enol'skii V Z, Kuznetsov V B and Tsiganov A V, 1994, J. Phys. A 27, 567.

[13] Kulish P P, Rauch-Wojciechowski S and Tsiganov A V,1996, J. Math. Phys. 37, 3463.

[14] Zeng Yunbo, 1997, J. Math. Phys. 38, 321.

[15] Zeng Yunbo, 1997, J. Phys. A 30, 3719.

[16] Zeng Yunbo and Ma W X, 1999, Physica A 274(3-4), 1.

[17] Dubrovin B A, 1981, Russian Math. Survey 36, 11.

[18] Its A and Matveev V, 1975, Theor. Mat. Fiz. 23, 51.

[19] Zeng Yunbo, 2000, J. Phys. A: Math. Gen. 33, L115.

[20] Antonowicz M and Rauch-Wojciechowski S, 1992, Phys. Lett. A 171, 303.

[21] Ragnisco O and Rauch-Wojciechowski S, 1992, Inverse Problems 8, 245.